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Newsvendor's Supply Chain Design Considering Non-Survival Risks

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Abstract:

Purpose: We consider the long-term survival of the newsvendor's supply chain (SC). In this regard, the main contribution of this paper is to identify and analyze, two risks arising from the random nature of the demand, a fundamental feature of the newsvendor problem (NVP). These risks include a decrease in demand caused by stock shortages and vendor bankruptcy due to insufficient inflows.

Design/methodology/approach: We present a mathematical model of the newsvendor's SC with three components: the Producer (P), the Vendor (V) and the Consumers (C). The model takes into account the relation between the components of the SC, the income and operative costs of P and V and the initial funds of V. We solve the model considering several assumptions related with the cost, the initial funds and the demand of C.

Findings: We identify and analyze two risks that threaten the continuity of the newsvendor's SC in the long term. The first comes from the decreasing in the demand derived from the low level of service that can result from optimizing the joint income of P and V without considering a level of service constraint, which has deserved little attention in the literature on the NVP. In fact, an unacceptable low service level conflicts with the assumption that demand remains stable over time. The second risk is that of V's bankruptcy due to insufficient monetary inflows, what can happen even if the expected value of V's daily net income is positive. The expected value of the daily unsatisfied demand and the probability of V's survival throughout a given time horizon are respectively proposed as indicators of the two risks mentioned. Procedures to calculate the indicators are described and illustrated with a numeric example.

Practical implications: First, in the design of the SC, under conditions similar to those of the NVP, a decision-maker must compare the options of resorting to V or not, considering the consequences on income and risk. Second, the objective of coordinating the SC, understood as the maximization of the joint income of P and V, has the limitation of not taking into account that the consumers, which are also part of the SC, can leave it when they deem that the service is unsatisfactory. Third, P must be aware, when setting the parameters of the SC, that its decisions determine those of V and their consequences; therefore, P must take into consideration the economic survival of V, in order to avoid disruptions in the SC and the negative consequences they imply for P itself.

Originality/value: Our approach is different from the most usual, which is focused on the criteria and the optimization procedures of the vendor. Instead, we consider the whole SC, including one essential component of it, namely, the consumers.

Keywords: newsvendor problem, supply chain design, supply chain risks, survival probability

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1. Introduction

The traditional version of the newsvendor problem (NVP) deals with a supply chain (hereafter, SC) consisting of three entities (the publisher, P, the vendor, V, and the consumers, C, who generate a random demand of a perishable product, which lasts only for one period, say a day). P decides and informs V of the price at which V may purchase the product and the price at which it will pay for the units that V returns at the end of the period. Then V, who is assumed to know C's probability law of demand, calculates how many units he should buy from P, at the beginning of the period, to maximize a function of its own daily net revenues (their expected value, for instance). This determines the sales, service level and income of P and V. There are also versions of the problem in which the SC consists of only two entities, P and C (P manufactures the product and undertakes its sale).

In the basic versions of the NVP the question consists in determining the size, q, of the daily order from V to P.

The randomness of the demand implies risks that can be more or less shared by P, V, and C. The literature dealing with this question sometimes refers to the risk of P, but mostly to that of V, basically in relation to the newsvendor's criteria for determining the value of q.

The approach taken in this work is different, because it is not concerned with the risk inherent to the dispersion of income in a stable SC, but with the risk that the SC cannot survive, taking into account that, when the SC have to work indefinitely (like, of course, that of the newspaper vendor) the concatenation of periods involves risks that, despite the fact that they may even involve the collapse of the SC, have been little studied.

We adopt two assumptions already present in the first formalization of the problem, namely, that both P and V know the distribution law of the demand, which is the same for all periods, and that the price at what V sells the product to C is a fixed market value.

It is also assumed that P's total revenue, i.e., the sum of that provided by the newsvendor SC and that coming from other sources (e.g. newspaper subscriptions) is sufficient to ensure the continuity of P, what implies that the collapse of the SC can only come from V and C. Therefore, this article focuses on two critical risks that threaten the survival of newsvendor's SC.

First, coordinating P and V may lead to a daily order size, q, that results in a very low service level for C, with fill rates that would be considered unacceptable in real-world inventory management. This situation can lead to a gradual decline in demand due to frequent stockouts. A high proportion of unsatisfied demand contradicts the assumption that the probability distribution of demand remains stable over time. In practice, this typically would lead to a loss of customers, who will turn to other suppliers or substitute products. Eventually, this can cause V to suffer from a lack of income, which may result in the collapse of the entire SC.

Second, the bankruptcy of V caused by insufficient net income, as a result of the fluctuations of the demand, even if its expected value would be high enough to ensure the newsvendor's survival. This is a very common phenomenon, the specialized media often echo the economic failures of franchised companies in various sectors of activity, such as clothing and fashion sales. Of course, the risk of bankruptcy depends on the criteria that V uses to determining the size of the daily order to P; however, once this criterion is known by P, the results, in the frame of the contract between P and V, depends exclusively on the values that P itself decides to give to the parameters of the SC.

The aimed contribution of the present paper is to analyze both mentioned risks in order to take them into account in the design of the SC. For this purpose, we propose indicators to quantify these risks and procedures to calculate them and we describe a procedure to consider the indicators in the design of the SC so that they reach the values desired by P.

The configuration of the rest of the paper is as follows. Section 2 presents a synthesis, focused on the indicated objective, of the state of the art on NVP. The specification of the model and the analysis of the behavior of the SC and its repercussions on its components (in particular on V and C) are dealt with in section 3. Section 4, which closes the article, includes the conclusions, with an outline of the steps of the decision process in the design of the SC of a perishable product with random demand, suggestions for prospective research, and managerial insights.

2. State of the Art

Although some authors trace the origins of the NVP to an article by Edgeworth from the 19th century (Edgeworth, 1888), others do not find any significant connection. NVP was introduced in the literature of inventory management by Arrow, Harris and Marschak (1951) and became popular, so to speak, in the academic world following the publication of Wagner's book (Wagner, 1969) on operations research (see Porteus, 2008, for a historical background of the problem).

Subsequently, it has given rise to a large number of publications, among which we mention only those most directly related to the objective of our research. In particular, we do not consider articles on pricing (affecting demand, through price changes or discounts), because the assumptions we adopt are incompatible with the possibility that P or V modify the market price, p.

Khouja (1999) reviews the literature, studies extensions of the NVP and makes suggestions for research. Porteus (2008) provides an excellent introduction to the NVP. Qin, Wang, Vakharia, Chen and Seref (2011) places the NVP in the framework of a SC of three elements and studies discount policies from P to V, and the repercussion on the volume of the order of the risk profile of V and of actions on the market. Choi (2012) includes sixteen chapters that study variants of the NVP.

The consideration of the NVP in the frame of SC management leads very naturally to focus on the question of the coordination. In this regard, Pasternack (1985) is particularly interesting because it considers channel optimization (which, in the case of NVP, comes to be the same as P and V coordination). He concludes that this can be achieved, even in an environment with multiple sellers with different distributions of the demand, with policies of admitting unlimited returns with partial credit, provided that the values of v (unit price at which P sells the product to V) and r (return unit price of the unsold units) satisfy a certain equation that is presented in the article and that does not depend on the distribution of demand. Each of the infinite pairs of values that satisfy the equation corresponds to a different distribution between P and V of the expected profit of the channel. It also shows that P's policy of not accepting returns is suboptimal and so is that of accepting unlimited returns for full credit. It also shows that policies allowing only partial returns are not appropriate in an environment of multiple sellers having different demand distributions, because then the optimal values of r and v depend on each seller's demand distribution.

Webster and Weng (2000), given the results of Pasternack (1985), introduces the concept of risk-free returns policies, which are those that, in spite of including the possibility of return, are exempt from risk for P, and with an expected value of P's profit not less than that corresponding to the no-returns policy, which is the most obvious in order to avoid the risk of P. Koulamas (2006), instead, proposes to achieve coordination with profit-sharing policies, without return, which require the ability of the manufacturer to monitor retail sales. Chen (2011) analyzes the revenue optimization of P with a no-returns policy and hence with deterministic revenue for P, determines v^*, the optimal value of v in this framework and compares this policy with those defined by a discount on the price v^* and admission of returns, which may be advantageous, relative to that of no-returns, both for the expected values of profits of P and V. Becker-Peth, Katok and Thonemann (2013) considers the case when the behavior of V is irrational but predictable.

Many papers have adopted assumptions different from that of V's risk-neutrality, especially after the introduction of prospect theory by Kahneman and Tversky (1979), whose impact on NVP is analyzed in Nagarajan and Shechter (2014) and Surti, Celani and Gajpal (2020). Yamini (2023) offers a literature review on the impact of loss aversion on the NVP, which is dealt with in Wang and Webster (2007, 2009), Xu, Wang, Dang and Ji (2017) and Wu, Bai and Zhu (2018). Keren and Pliskin (2006) and Katariya, Cetinkaya and Tekin (2014) consider risk aversion in the NVP, the last comparing risk-neutral and risk-averse newsvendor. Choi, Li and Houmin (2008) applies mean-variance analysis; Xu, Meng, Shen, Jiang and Ji (2015) and Chen (2024) the Conditional Value at Risk (CVaR) approach.

Chen (2024) uses also CVaR and points out that the objective of maximizing this criterion is in conflict with that of having a satisfactory fill rate. This is very interesting, because manifests that some approaches of the NVP show scarce connection with the usual theory and practice of inventory management. In fact, maximizing the utility criteria of V may result in unacceptable values of the fill rate and, thus, is surprising that this issue had deserved so little attention so far.

Therefore, we can conclude from this review of the literature that:

On the one hand, there is abundant research work on the way in which V can deal with risk, within a framework set by P. But little attention has been devoted to the way in which P's decisions impact the risk of V and, finally, in that of the whole SC.

And, on the other hand, there is a need to integrate the consideration of the fill rate in the design of the newsvendor SC.

The aim of the present paper is to contribute to fill these just now mentioned gaps.

3. The model and an illustrative example

Presented with current terminology, the NVP is considered, as indicated above, in the frame of a SC with three components:

P: the publisher or in general the manufacturer of a perishable product (such as printed newspapers, bakery, fashion or holiday products…) with a useful life of one period (which, WLOG, we usually will refer to as one day). P is risk-neutral and can produce and make available at the beginning of every day any amount of the product, with a variable unit production cost c monetary unit (hereafter, *MU*), a market price p *MU* and a null residual value at the end of the period. P can choose to sell the product directly to the market, without intermediaries (with a fixed cost per period equal to K *MU*) or to use an intermediary, V.

V: the vendor (perhaps a single newsvendor; however, it can also be a set of vendors with disjoint business areas, but we will not consider this possibility). V, which is risk-neutral, can make at the beginning of each period (i. e., before observing the actual demand), a single order to P for the number of units, q, it considers convenient that P will sell to it at a price v *MU*, where $c < v < p$. V determines the size of the order for the purpose to optimize the expected value of its utility, assimilated here to the expected value of the corresponding income. V has a daily fixed cost equal to k *MU* and has no variable costs (that is, those that depend on the volume of sales).

C: the consumers, who present a finite random demand, with a probability distribution known to P and V. V sells daily to C, at the price p, determined by the market, a number of units that is the minimum between q and the total demand from C, since V does not have the possibility to make a complementary order to P in the course of the day. If P so stipulates, V may return to P, at the end of the period, the unsold units and receive a return unit price, r *MU*, such that $r < v$. We therefore limit ourselves to policies with unlimited returns paid from P to V below the price v.

Sometimes disposal costs are also considered, but we dispense with them for the sake of simplicity and because that they would not contribute anything substantial to the consideration of the risks that are the subject of our article.

The daily demand is represented by a random continuous variable, x, with a probability density function, $h(x)$, continuous, such that $h(x)>0$ for $x_m \leq x \leq x_M$ and $h(x) = 0$ elsewhere, known by P and V, with an expected value $\bar{x}$;

let $H(x)$ the corresponding distribution function. Obviously, in practice x_M is finite, although, if appropriate, in the model one can deal with a law such that $h(x) > 0$ for $x_m \leq x < \infty$.

We do not consider here costs or benefits derived from the elimination or recycling of unsold units, nor costs for unsatisfied demands due to lack of stock (shortage costs). Although shortage cost is a very present parameter in the inventory management literature and can easily be introduced in the model (see, for example: Pasternack, 1985; Chen, 2011), in practice it is difficult to determine its value, so we have preferred to take into consideration stockouts by way of service level, measured as the expected value of the daily number of units not sold due to lack of stock, $U(q)$ or that of the fill rate $\varphi(q) = 100 \cdot S(q)/\bar{x}$, where $S(q)$ is the expected value of the daily sales.

We will assume that P cannot go bankrupt because it has other safe and sufficient sources of income, other than those coming from the SC considered, but that V only has a fund of F MU for contingencies and no other income that the corresponding to sales to C. So, V can go bankrupt, given the random nature of the demand, unless its income has a high enough lower bound.

Under the indicated assumptions, it is easy to adapt the results that can be found in several papers and particularly in Pasternack (1985) and that, therefore, do not need to be justified here.

Let q represent the number of units of the product available for sale each day, regardless of whether P sells directly or through V. The expected value of the number of units sold in a period, denoted as $S(q)$, is:

$$S(q) = \int_{x_m}^{q} x \cdot h(x)\, dx + q \cdot \int_{q}^{x_M} h(x) dx = \int_{x_m}^{q} x \cdot h(x) dx + q \cdot [1 - H(q)] \quad (1)$$

Therefore, the expected value of the number of unsold (and therefore returned) units, denoted as $R(q)$, is:

$$R(q) = q - S(q) \quad (2)$$

And that of the unsatisfied demand, denoted as $U(q)$:

$$U(q) = \int_{x_m}^{x_M} x \cdot h(x)\, dx - S(q) = \bar{x} - S(q) \quad (3)$$

Table 1 shows the notation that has been defined up to this point and other that will be used later.

P	The publisher|producer, risk-neutral.
c	Unit variable production cost of the item.
p	Market price of the item.
K	Fixed daily cost assumed by P if it decides to sale directly the item to the market, without intermediaries.
V	The vendor, risk-neutral.
v	The price at which P sells the item to V.
r	The price at which P buys to V the unsold units of each period.
q	The number of units that P or V put for sale daily.
k	Fixed daily cost of V.
C	The consumers (the market).
x	Continuous random variable corresponding to the daily demand from C to V ($x_m \leq x \leq x_M$).
$h(x)$	Probability density function of x.
$\bar{x}$	Expected value of x.
$H(x)$	Cumulative distribution function of x.

$S(q)$	Expected value of the daily sales.
$R(q)$	Expected value of the daily unsold units, which V returns to P.
$U(q)$	Expected value of the daily unsatisfied C's demands.
$\hat{U}$	Maximum admissible value for the expected unsatisfied daily demands
$\varphi(q)$	Fill rate, expected value of the ratio between unsatisfied and satisfied demands from C to V, expressed usually as a percentage $\varphi(q) = 100 \cdot S(q)/\bar{x}$.
$E(q)$	Expected value of the daily gross income of P if P sells the product directly to C.
$\tilde{E}(q) = E(q) - K$	Expected value of the daily net income of P if P sells the product directly to C.
q^*_p	Value of q that maximizes the expected value of P's daily income if P sells the product directly to C.
$E(r,v,q)$	Expected value of the daily income of P if the product is sold by V.
$e(r,v,q)$	Expected value of the daily gross income of V if the product is sold by V.
$\tilde{e}(r,v,q) = e(r,v,q) - k$	Expected value of the daily net income of V if the product is sold by V.
$\sum(q)$	Expected value of the daily net income of the SC if the product is sold by V.
q^*_v	Value of q that maximizes the expected value of V's daily income.
q^*_{sc}	Value of q that maximizes the expected value of the SC's daily income.
$\hat{e}$	Target value for the expected value of the daily gross income of V.
$\hat{r}$	Highest value of r so that e is no lower than $\hat{e}$.
ρ	The ratio $(p - v)/(p - r)$.
F	Initial reserve of V's funds.
T	Periodicity (number of days) of the payments from V to P and, also, of the payments of the fixed cost of V.
n	Number of cycles of T days.
$s(T,n)$	Survival probability of V after n cycles of T days.
$\hat{s}$	Minimum admissible value of $s(T,n)$.

Table 1. Notation used in this section and the following ones

To illustrate with an example the developments that follow we will use the data indicated below:

$c = 0.3$ MU, $p = 1.5$ MU, $h(x) = 0.002$ $(0 \leq x \leq 500)$, $K = 10$, $k = 5$

The fact that $h(x)$ is a uniform law $U[0,x_M]$ (i.e., $h(x) = 1/x_M$, $0 \leq x \leq x_M$; $h(x) = 0$, $x>x_M$) makes easier calculations: $H(x) = x/x_M$, $0 \leq x \leq x_M$; $H(x) = 1$, $x>x_M$; $\bar{x} = x_M/2$; $H^{-1}(x) = x_M \cdot x$. In the example, $\bar{x} = 250$, $H^{-1}(x) = 500 \cdot x$.

3.1. The Case of Direct Sale From P to C and the Risk Derived from Unsatisfied Demand

In this case (SC with two components) P's income is random and it must assume the fixed cost K.

The expected value of P's income is:

$$E(q) = p \cdot S(q) - c \cdot q \tag{4}$$

And, deducting the fixed costs, K, we get the expected value of net income:

$$\tilde{E}(q) = E(q) - K = p \cdot S(q) - c \cdot q - K \tag{5}$$

The value of q that maximizes the income is deduced from the zero-derivative condition, which, due to the concavity of the functions, is necessary and sufficient for maximum. As this gives $\frac{p-c}{p} = H(q^*_p)$, we have:

$$q_P^* = H^{-1}\left(\frac{p-c}{p}\right) \tag{6}$$

With the data of the example, it results $q^*_p = H^{-1}(0.8) = 400$, and, according respectively to eqs. (1), (2), (3) and (5): $S(q^*_p) = 160+400 \cdot 0.2 = 240$, $R(q^*_p) = 400\text{-}240 = 160$, $U(q^*_p) = 250\text{-}240 = 10$, $\tilde{E}(q^*_p) = 1.5 \cdot 240\text{-}0.3 \cdot 400\text{-}10 = 230$; and, in accordance of this definition in Table 1, $\varphi(q^*_p) = (240/250) \cdot 100 = 96.00$,

If the fill rate is too low (as that of the example would might be in practice), the assumption that the demand density function remains unchanged over time is unrealistic.

If P considers that the value of $U(q^*_p)$ is too high, it must increase the value of q, what implies a reduction in the value of $E(q)$.

If the admissible value for the expected unsatisfied daily demands is $\hat{U}$, a value $(\hat{q}^*_p)$ must be found such that $S(\hat{q}^*_p) = \bar{x} - \hat{U}$.

In the example, with $\hat{U} = 1$, the result is $S(q^*_p) = 249$. From which we deduce $\hat{q}^*_p = 468.38$. By imposing the condition $\hat{U} \leq 1$ the value of E goes from 240 MU, when the value of U is not restricted, to 232.99 MU.

Note that $\hat{U} = 0$ can only be attained, with $q = x_M$, if the demand has a finite maximum value.

3.2. The Case of Sale Through V and the Risk of the Newsvendor's Bankruptcy

If the product reaches C via V, the expected values of the revenues of P and V, disregarding V's fixed costs, are, respectively

$$E(r,v,q) = (v-c) \cdot q - r \cdot R(q) \tag{7}$$

$$e(r,v,q) = p \cdot S(q) + r \cdot R(q) - v \cdot q \tag{8}$$

And therefore, the expected value of the daily gross income of the SC is:

$$\Sigma(q) = E(q) + e(q) = p \cdot S(q) - c \cdot q \tag{9}$$

Which coincides with the expected value of the daily gross income of P in the case analyzed in 3.1 (if we ignore the fixed costs, the monetary exchanges between P and V, which take place inside the SC, have no repercussions on the overall result).

Note that, unlike Σ, which depends only on q, E and e depend on r and v as well. In other words, the value of q determines the overall result of the SC, but the distribution of this result between P and V depends on the pair (r,v).

It should be noted that the expected value of V's available funds as a result of daily operations is:

$$\tilde{e}(r,v,q) = p \cdot S(q) + r \cdot R(q) - v \cdot q - k \tag{10}$$

The optimal values of q for P, V, and the SC are deduced from the zero-derivative condition, which due to the concavity of the functions, is a necessary and sufficient condition for maximum:

$$q_P^* = H^{-1}\left(\frac{v-c}{r}\right),\ q_V^* = H^{-1}\left(\frac{p-v}{p-r}\right), q_{SC}^* = H^{-1}\left(\frac{p-c}{p}\right) \tag{11}$$

And it must be borne in mind that the value of q is determined by V, based on what it considers to be data, among which the values of v and r which for P are decision variables. Although apparently it is V who determines q, actually V's behavior is the result of P's decisions. Therefore, P may predict the value of q^*_v, provided that V knows $h(x)$ and is able to determine the optimum value of q for the adopted criterion (in the present paper, the expected value of the daily income).

The value of q^*_v (and thus the overall SC result) depends on the pair (r,v), but it is the same for all pairs that yield the same result for $\rho = (p-v)/(p-r)$. Instead, the distribution between P and V of the SC income is different for each pair (r,v).

The best possible result for the tandem P-V as a whole is obtained when:

$$q^*_V = q^*_{SC} \tag{12}$$

Or, equivalently:

$$\frac{p-v}{p-r} = \frac{p-c}{p} \tag{13}$$

By resorting to the intermediary V, P saves the fixed costs K, but continues to have random income unless it admits no returns ($r = 0$). When this policy is adopted, E, e, and Σ depend only on v and the daily number of units put on sale is:

$$q^*_V = H^{-1}\left(\frac{p-v}{p}\right) \tag{14}$$

Then P, to optimize its income, must solve the following problem:

$$\max_{c \le v \le p} E(q^*_V) = (v-c) \cdot q^*_V = (v-c) \cdot H^{-1}\left(\frac{p-v}{p}\right)$$

in relation to which, Chen (2011) shows that, with $h(x)$ continuous, there exists a single optimal value of v, which, if $h(x)/H(x)$ is decreasing, is $<p$.

However, since the maximum of $\Sigma(q)$ is reached for $q^*_{sc} = H^{-1}((p-c)/p)$, it is clear that, with $r = 0$, the optimal value of $\Sigma(q)$ will be different and, therefore, less than the one corresponding to the P-V coordination since $(p-c)/p = (p-v)/p$, unless $v = c$, with which the SC would be infeasible because P's income would be nil.

In the case of the example, if the sale is made via V, without return ($r = 0$), with $v = c$ we have $q^*_v = H^{-1}(0.8) = 400$. P and V are coordinated, with $S(q^*_v) = 240$, $E(q^*_v) = 0$, $e(q^*_v) = 240$, $\Sigma(q^*_v) = 240$, $R(q^*_v) = 160$, $U(q^*_v) = 10$. That is, the SC is inviable from the point of view of P, as indicated above.

However, with $r = 0$ the optimal value of v for P, is 0.9, from which $q^*_v = H^{-1}(0.4) = 200$, $S(q^*_v) = 160$, $E(q^*_v) = 120$, $e(q^*_v) = 60$, $\Sigma(q^*_v) = 180$, $R(q^*_v) = 40$, $U(q^*_v) = 90$, $\varphi(q^*_v) = 64.00$.

In this example, the no-returns policy implies a 25% reduction in SC results, relative to those obtainable with the P-V coordination, and the expected value of unserved demands is multiplied by 9, with a clearly unacceptable fill rate. P's expected income, although K is spared, has been reduced to slightly more than half of P's net income under the direct sale option. The only advantage for P is that its income is deterministic.

That is, as established in Pasternack (1985), the policy of not admitting returns is suboptimal and gives rise to a result for the total SC lower than that obtained if P and V are coordinated. And the consequence is (Koulamas, 2006) the double marginalization, i.e. the fact that P and V obtain a lower income than what they could achieve with a policy of P that coordinated these two components of the SC.

If P adopts an unlimited return acceptance policy, with $0 < r < v$, the coordination of P and V is achieved when $(p-c)/p = (p-v)/(p-r)$, equivalent to:

$$v = \frac{p-c}{p} \cdot r + c = \rho^*_P \cdot r + c \tag{15}$$

Where $\rho^*_p = (p-c)/p$ (in the example, $v = 0.8 \cdot r + 0.3$). With this condition, $q^*_v = q^*_p$ and, without considering the fixed costs, the total result of the SC is the same as the optimum corresponding to the case of the SC with two elements.

The distribution of this result between P and V, depends on the values of the pair (r,v) linked by the equation $v = \rho^*_p \cdot r + c$, from which it results:

$$E = [S(q) - (1 - \rho_P^*) \cdot q] \cdot r \tag{16}$$

And since, with the P-V coordination, the value of $q = H^{-1}(\rho^*_p)$ and consequently the value of S do not depend on r, E is a linear function of r, regardless of the function $h(x)$:

$$E = \alpha \cdot r \; (0 < r < p) \tag{17}$$

With $a = S[H^{-1}(\rho^*_p)] - (1\text{-}\rho^*_p) \cdot H^{-1}(\rho^*_p)$. Therefore:

$$e = \Sigma - E = \Sigma - \alpha \cdot r \tag{18}$$

Where Σ, with the P-V coordination, does not depend on r.

So, $E \longrightarrow 0$ as $r \longrightarrow 0$, and $E \longrightarrow \Sigma$ as $r \longrightarrow v \longrightarrow p$, while e has the opposite behavior. The SC is not viable for $r = p$ nor for $r = 0$ because, respectively, the revenues of V and P are zero.

Given these expressions, is straightforward to determine the highest value of r, $\hat{r}$, so that e is no lower than a given value, $\hat{e}$.

It is worth noting that the latter developments and, specifically, the linear dependence of E and e with respect to r will remain true as long as v i r satisfy an expression of the form $v = \rho \cdot r + p \cdot (1\text{-}\rho)$, since this implies that $(p\text{-}v)/(p\text{-}r) = \rho$ and that the pairs (r,v) satisfying this relation lead to the same value of q. If (for example, in order to guarantee a certain service level) the condition $q = \hat{q}$ is imposed, this can be achieved with $v = \hat{\rho} \cdot r + p \cdot (1\text{-}\hat{\rho})$, where $\hat{\rho} = H(\hat{q})$.

It is striking that high values of r, which seem to be favorable to V, when P and V are coordinated actually imply low values of e, because, for a given q, v grows with r. And if P optimizes its own income quickly drags V into ruin. Therefore, P has to settle to obtaining at most an expected value of its income equal to $\Sigma\text{-}k$, which is equivalent to ensuring V an expected value of gross income $e = k$ (therefore, an expected value of the net income equal to 0). However, given the random nature of the income of V, this does not ensure its survival, as discussed below.

In the example, without the service level condition, $E = 160 \cdot r$, $e = 240\text{-}160 \cdot r$; with $\hat{e} = k = 5$ it results $r \leq \hat{r} = 1{,}46875$, $v \leq 1{,}475$.

Given the random nature of demand, V has a risk of bankruptcy that depends on the distribution of demand and the parameters that shape the system. In the example considered, the probability that any day V does not have an income equal to or greater than 0, with values of r and v that coordinate P and V ($v = 0.8 \cdot r + 0.3$) is 0.16. With the same assumption, the probability that the income does not exceed the value of $k = 5$ goes from 1/6 for $r = 0$ to 0.8 for $r = 1.484375$ (and is equal to 1 for higher values).

However, in general determining the risk of bankruptcy is more complex, because it depends not only on k, but also on the amount, F, the reserve of funds of V, on the periodicity of liquidations from V to P, and of the payment of the daily fixed costs, k.

If the fixed sales do not ensure a sufficient income, in general the random nature of the demand implies the risk of bankruptcy of V.

We propose as a criterion, in relation to this risk, that the survival probability of V, after a certain number of days, must be greater than or equal to a certain threshold $\hat{s}$.

We will assume that the periodicity of the payments from V to P and of the payment of the daily fixed costs, k, is equal, in both cases, to T days. And it must be determined the minimum value of the expected value of the gross income of V, $\hat{e}$, so that the survival probability of V after n cycles of T days, $s(T,n)$, satisfies the indicated criterion.

Once $\hat{e}$ is determined, it is immediate to calculate the pair (r,v) that ensures this value of the expected value of V's income.

To calculate $\hat{e}$ it must be taken into account that the probability law of the gross income of V has the peculiarity of being continuous in the interval $[(p\text{-}r)\cdot x_m\text{-}(v\text{-}r)\cdot q,(p\text{-}v)\cdot q)$, with a value of $1\text{-}H(q)$ when it reach the value $(p\text{-}v)\cdot q$. For this reason, unless T is very large, the approximation to the normal distribution of the sum of the incomes of the T days comprising a settlement cycle is not appropriate and the specific development shown in the Appendix is required (the Appendix also includes the application to the case that the demand follows a uniform distribution).

Thus, it has been calculated, with the data of the example and $r = 1.46875$ (which implies $v = 1.4727$ and $e = 5$), the probability law of the gross income of V generated during a cycle of $T = 7$ days (which is shown in Figure 1). Moreover, the survival probability of V has been calculated, with $F = 0$, as a function of n, the number of cycles of 7 days elapsed since the start of its activity (Figure 2), and the probability law of the funds of V available after 10 weeks, in case it has survived the previous weeks (Figure 3).

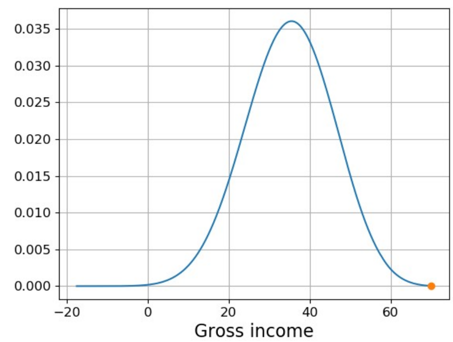

Figure 1. Probability law of the gross income of V generated during a cycle of 7 days, with $x_m = 0$, $x_M = 500$, $k = 5$, $F = 0$, and values of r, v, q satisfying $e(r,v,q) = k$

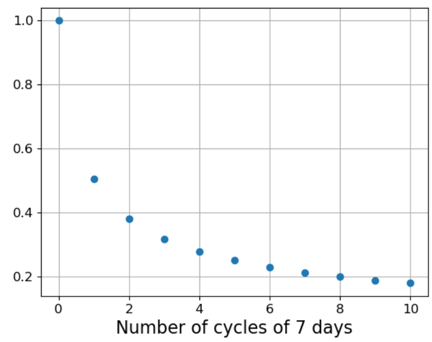

Figure 2. Survival probability of V as a function of the number of cycles of 7 days elapsed since the start of its activity, with $x_m = 0$, $x_M = 500$, $k = 5$, $F = 0$, and values of r, v, q satisfying $e(r,v,q) = k$

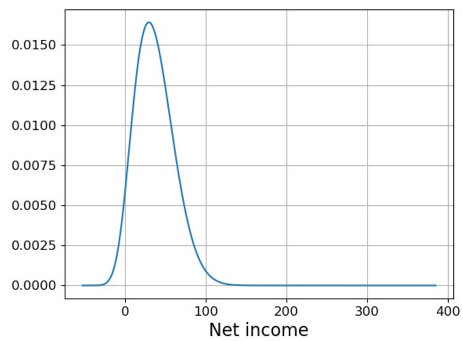

Figure 3. Probability law of the funds of V available after 10 cycles of 7 days, in case it has survived the previous cycles, with $x_m = 0$, $x_M = 500$, $k = 5$, $F = 0$, and values of r, v, q satisfying $e(r,v,q) = k$

The value of $\hat{e} \mid s(T,n) \geq \hat{s}$ must be found iteratively, either by progressively increasing the value of e or by means of a binary search, for example. Figure 4 shows the values of $s(7,10)$ for several values of e, with $k = 5$ and $F = 0$. For $\hat{s} = 0.99$ and $\hat{s} = 0.999$, the resulting values for $\hat{e}$ are, respectively, 18.16 and 80.80.

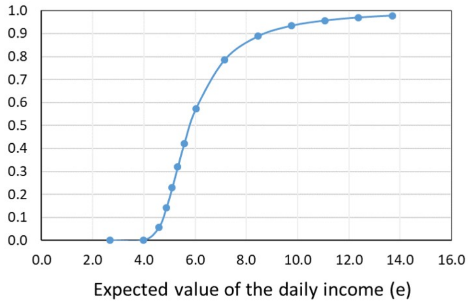

Figure 4. Survival probability of V after 10 cycles of 7 days as a function of the expected value of the daily gross income of V, $e(r,u,q)$, with $x_m = 0$, $x_M = 500$, $k = 5$, and $F = 0$

Alternatively, the value of $\hat{e}$ corresponding to a given $\hat{s}$, can be obtained by simulation (in short, it is a matter of estimating, as a proportion, $s(T,n)$).

Additionally, some sensitivity analysis has been done to studying how the survival probability depends on the length of the cycle, T, on the fixed daily cost of V, k, and on the variance of the demand. In all cases, the values of r, u, q assure that the expected value of the daily gross income of V, $e(r,u,q)$, is equal to k.

Figure 5 shows the survival probability during the first 30 days of activity for $T = 1,2,\ldots,7$, with $k = 5$, and $F = 0$.

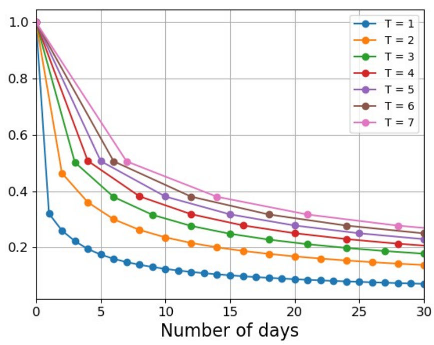

Figure 5. Survival probability of V during the first 30 days of activity for $T = 1,2,\ldots,7$, with $x_m = 0$, $x_M = 500$, $k = 5$, $F = 0$, and values of r, u, q satisfying $e(r,u,q) = k$

As for the fixed daily cost of V, several values of k has been considered, with initial reserve of V's funds, $F = 0.01 \cdot k, 0.1 \cdot k, \ldots, 5 \cdot k$. With this assumptions, the calculations show that survival probability does not depend on k but, as expected, it does on F/k. Figure 6 shows the survival probability after 4 cycles of 7 days for $k \in [5,100]$ and $F/k \in [0,5]$.

As for the variance, several probability laws of the demand has been considered, all of them continuous uniform with different values for x_m and x_M and with the same expected value ($(x_m+x_M)/2 = 250$). In the case that $F = 0$, we obtained that the survival probability does not depend on x_m and x_M. For $F>0$, Figure 7a and Figure 7b show for $F/k = 0.01$ and $F/k = 1$, respectively, the survival probability for $x_m \in \{0, 50, 100, 150, 200, 225, 245\}$ with $T = 7$, $k = 5$.

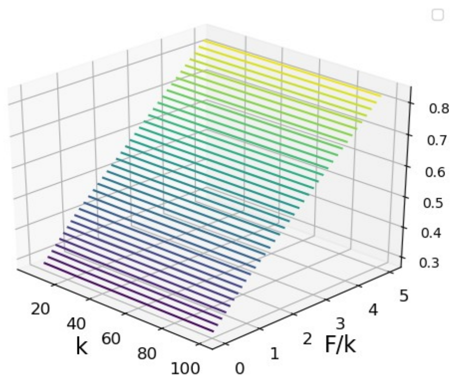

Figure 6. Survival probability of V after 4 cycles of 7 days, for several values of k and F/k, with $x_m = 0$, $x_M = 500$, $k = 5$, $F = 0$, and values of r, v, q satisfying $e(r,v,q) = k$

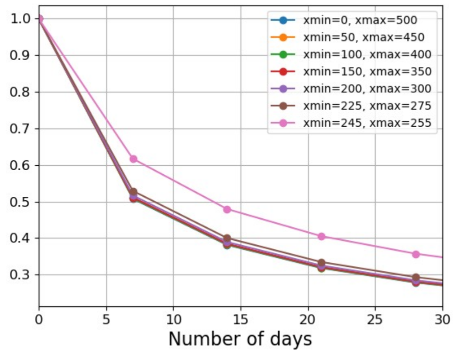

Figure 7a. Survival probability of V for several values of x_m and x_M with $T = 7$, $k = 5$, $F = 0.05$ and values of r, v, q satisfying $e(r,v,q) = k$

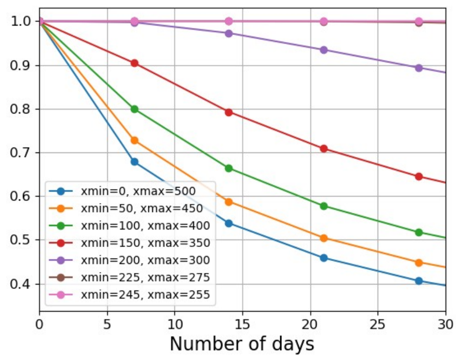

Figure 7b. Survival probability of V for several values of x_m and x_M with $T = 7$, $k = 5$, $F = 5$ and values of r, v, q satisfying $e(r,v,q) = k$

4. Conclusions on the SC Design Process for the NVP and Managerial Insights

This article adopts an approach to the NVP that differs markedly from the standard one, which typically emphasizes the vendor's criteria and optimization procedures in a given setting.

Instead, we focus on the design of the SC under the assumptions that define the NVP, taking into account one of its essential components: the consumers. This point of view come to the light two risks that pose long-term threats to the continuity of the newsvendor's SC.

The first of the two mentioned risks comes from the decreasing in the demand derived from the low level of service that can result from optimizing the joint income of P and V without considering a level of service constraint, which, although it is an essential characteristic of any system of inventory management, has deserved little attention in a substantial portion of the literature on the NVP. In fact, an unacceptable low level of service is hardly compatible with the assumption that the law of the demand does not change throughout the time.

The second risk is that of V's bankruptcy due to insufficient monetary inflows, what can happen even if the expected value of V's daily net income is positive.

Indicators are proposed to measure these risks and procedures to calculate them.

The results shown in the previous section suggest, to tune the SC, the sequence of steps shown in Figure 8.

Data: $c, p, h(x), \widehat{U}, \hat{s}$

Calculate $q_P^*, U(q_P^*)|\varphi(q_P^*), \hat{q} \leftarrow q_P^*, \hat{\rho} \leftarrow \rho_P^*$

If $U(q_P^*) > \widehat{U}$, calculate $\hat{q}\ |U(\hat{q}) = \widehat{U}$

calculate $\hat{\rho}|H^{-1}(\hat{\rho}) = \hat{q}, \tilde{E}(\hat{q})$

Calculate, with $v = \hat{\rho} \cdot r + p \cdot (1 - \hat{\rho})$, $\hat{e}|s(T.n) = \hat{s}$

calculate $\hat{r}|e(\hat{r}) = \hat{e}, E(\hat{r})$

Figure 8. Calculations for SC tuning

The decision between SC with or without V must take into account the results obtained with the calculation process summarized in Figure 8. It is clear that the specific criteria for choosing one or another option depend on the characteristics of P, V and C.

It is clear that the adoption, in this article, of some of the most common assumptions in the definition of NVP imply limitations in some of our proposals, although the qualitative considerations on the survival risks of the SC are essentially derived from the randomness of demand. In general, the probability law of demand will not be known a priori, but can be inferred from the available market data in successive periods. Other assumptions, of course, may be incompatible with specific real situations, which suggests lines of research aimed at expanding the scope of our findings. Specifically, the extension of the calculations to the assumption that P or V, or both, are not risk-neutral or agree to different types of contracts than that assumed in this article. Also, on the impact of the level of service on the evolution of the law of demand, on the procedures that P can arbitrate to avoid the bankruptcy of V in the event that there is a succession of numerous periods of low demand, on the criteria to deciding between selling directly to market or contracting V as an intermediary between P and C and on the sensitivity analysis of the survival probability of V with respect to the variance, considering different demand probability laws. Also, as it may be the case that P has monopolistic power and can influence the selling price of the product to C, a particularly challenging line of research consists of including this assumption in the design process of the SC considered in our article.

Our analysis suggests some managerial insights as well. First, that in the design of the SC, under conditions similar to that of the NVP, the decision maker must compare the options of resort or not to V, considering the consequences on income and risk. Second, the objective of coordinating the SC, understood as the maximization of the joint income of P and V, has the limitation of not taking into account that the consumers, which are also part of the SC, can leave it when they deem that the service is unsatisfactory. Third, P must be aware, when setting the parameters of the SC, that its decisions determine those of V and their consequences; therefore, P must take into consideration (when setting the values of *v*, *r*, and T) the economic survival of V, in order to avoid disruptions in the SC and the negative consequences they imply for P itself.

It could be said that randomness disorients the invisible hand and that, consequently, a global consideration of SC is needed in order to ensure its survival and, with this restriction, the best possible results.

Declaration of Conflicting Interests

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Appendix: Computing V's Risk of Failure

The survival probability of V after a cycle of T days is equal to the probability that its available funds at the end of this period are not negative.

To calculate this probability, we first determine F_V, the cumulative distribution function (CDF) of the income of V generated in a single day. The income of V in a single day, y, is a random variable with the following values

$$y = \begin{cases} (p-v)\cdot q - (p-r)\cdot(q-x), & x < q \\ (p-v)\cdot q, & x \geq q \end{cases} \quad \text{(A.1)}$$

Where x is a random variable such that $x_m \leq x \leq x_M$ with a known CDF H. So,

$$F_V(Y) = \begin{cases} 0 & Y < y_m \\ H\left(\dfrac{Y+(v-r)\cdot q}{(p-r)}\right) & y_m \leq Y < y_M \\ 1 & Y \geq y_M \end{cases} \quad \text{(A.2)}$$

Where $y_m = (p\text{-}r) \cdot x_m - (v\text{-}r) \cdot q$ and $y_M = (p\text{-}v) \cdot q$. Observe that F_V is a function with a jump discontinuity at the point $(p\text{-}v) \cdot q$ and that size of the jump is 1-$H(q)$.

Next, we calculate $F_{V,T}(Z)$, the CDF of z, the income of V generated during a cycle of T days, which are $\sum_{i=1}^{T} y_i$ with y_i random variables of the income of V in a single day. Given that each y_i can take the value y_M with probability 1-$H(q)$ and lower values with probability $H(q)$, the probability that t of these variables take a value less than y_M and the remaining $T\text{-}t$ take the value y_M is $\binom{T}{t}\cdot H(q)^t\cdot(1-H(q))^{T-t}$. Then, the CDF of the income of V in a cycle T days, $F_{V,T}(Z)$ is

$$F_{V,T}(Z) = \begin{cases} \displaystyle\sum_{t=1}^{T} F_{V<,t}\left(Z-(T-t)\cdot y_M\right)\cdot\binom{T}{t}\cdot H(q)^t\cdot(1-H(q))^{T-t} & Z < z_M \\ 1 & Z \geq z_M \end{cases} \quad \text{(A.3)}$$

Where $z_M = T\,y_M$, $F_{V<,t}(x) = p(\sum_{i=1}^{t} y_{<,i} \leq x)$ and $y_{<,i}$ is a continuous random variable which takes values between y_m and y_M, with CDF $F_{V<}(Y) = H((Y+(v\text{-}r)\cdot q)/(p\text{-}r))/H(q)$ for $y_m \leq Y \leq y_M$, 0 for $Y < y_m$ and 1 for $Y > y_M$, and probability density function $f_{V<}(Y) = h((Y+(v\text{-}r)\cdot q)/(p\text{-}r))/H(q)$ for $y_m \leq Y \leq y_M$ and 0 elsewhere. We note that $F_{V<,t}(x)$ equals 0 for $x \leq t \cdot y_m$ and equals 1 for $x \geq t \cdot y_M$. Moreover, $F_{V,T}(Z) = 0$ for $Z \leq T \cdot y_m$, $F_{V,T}(Z)$ is continuous for any value of Z except z_M and the size of the jump at z_M is $(1\text{-}H(q))^T$. Let us define

$$f_{V,T}(Z) := \sum_{t=1}^{T} f_{V<,t}(Z-(T-t)\cdot y_M)\cdot\binom{T}{t}\cdot H(q)^t\cdot(1-H(q))^{T-t} \tag{A.4}$$

For all Z, where $f_{V<,t}$ is the probability density function of $\sum_{i=1}^{t} y_{<,i}$.

The funds of V available at the end of the first cycle of T days are $a_1 = z_1 + F - k\cdot T$, where F is the amount of the funds available at the beginning of the cycle, z_1 the income of V generated during the cycle and k the daily fixed costs. The CDF of a_1 is $F_1(A_1) = F_{V,T}(A_1 + k\cdot T - F)$ for all A_1. Let's define

$$f_1(A_1) := \left.\frac{dF_1(x)}{dx}\right|_{x=A_1} = f_{V,T}(A_1 + k\cdot T - F) \tag{A.5}$$

For $F + z_m - k\cdot T \leq A_1 < F + z_M - k\cdot T$, $f_1(A_1)$: = 0 for $A_1 < F + z_m - k\cdot T$ and for $A_1 = F + z_M - k\cdot T$, we extend the function by continuity.

The survival probability of V at the end of the first cycle of T days is $s(T,1) = 1\text{-}F_1(0)$.

Let's $S(T,n)$ denote the survival condition of V after n cycles of T days.

In case of survival, the conditional CDF of available funds at the end of the first cycle is

$$F_1(A_1|S(T,1)) = \begin{cases} 0 & A_1 < 0 \\ \dfrac{F_1(A_1)-F_1(0)}{1-F_1(0)} & A_1 \geq 0 \end{cases} \tag{A.6}$$

The funds of V available at the end of the second cycle of T days are $a_2 = z_2 + a_1 - k\cdot T$, where z_2 is the income of V generated during the second cycle, and its CDF is

$$\begin{aligned} &F_2(A_2|S(T,1)) \\ &\quad= \int_0^{F+z_M-k\cdot T} F_{V,T}(A_2+k\cdot T-x)\cdot\frac{f_1(x)}{1-F_1(0)}\,dx \\ &\quad+ F_{V,T}(A_2+2k\cdot T-F-z_M)\cdot\frac{(1-H(q))^T}{1-F_1(0)} \end{aligned} \tag{A.7}$$

For all A_2. Observe that $F_2(A_2|S(T,1)) = 0$ for $A_2 \leq z_m - k\cdot T$, $F_2(A_2|S(T,1)) = 1$ for $A_2 \geq F + 2\cdot(z_M - k\cdot T)$, the function is continuous for any value of A_2 except $F + 2\cdot(z_M - k\cdot T)$ and the size of the jump at that point is

$$\frac{(1-H(q))^{2\cdot T}}{1-F_1(0)} \tag{A.8}$$

We define

$$\begin{aligned} f_2(A_2) &:= \left.\frac{dF_2(x|S(T,1))}{dx}\right|_{x=A_2} \\ &= \int_0^{\min\{F+z_M-k\cdot T,\,A_2+k\cdot T-z_m\}} f_{V,T}(A_2+k\cdot T-x)\cdot\frac{f_1(x)}{1-F_1(0)}\,dx \\ &+ f_{V,T}\big(A_2+k\cdot T-F-2\cdot(z_M-k\cdot T)\big)\cdot\frac{(1-H(q))^T}{1-F_1(0)} \end{aligned} \tag{A.9}$$

For all $A_2 < z_M - k\cdot T$.

$$f_2(A_2) := \left.\frac{dF_2(x|S(T,1))}{dx}\right|_{x=A_2} = (1-H(q))^T \cdot \frac{f_1(A_2+k\cdot T-z_M)}{1-F_1(0)} + \int_{A_2+k\cdot T-z_M}^{\min\{F+z_M-k\cdot T, A_2+k\cdot T-z_m\}} f_{V,T}(A_2+k\cdot T-x)\cdot\frac{f_1(x)}{1-F_1(0)}dx + f_{V,T}(A_2+k\cdot T-F-2\cdot(z_M-k\cdot T))\cdot\frac{(1-H(q))^T}{1-F_1(0)} \quad (A.10)$$

For all $z_M - k\cdot T \leq A_2 < F + 2\cdot(z_M - k\cdot T)$ and for $A_2 = F + 2\cdot(z_M - k\cdot T)$, we extend the function by continuity.

The non-survival probability of V at the end of the second cycle is $\bar{s}(T,2) = s(T,1)\cdot F_2(0|S(T,1))$ and the survival probability, $s(T,2) = s(T,1)\cdot(1-F_2(0|S(T,1)))$. In case of survival at the end of the second cycle, the conditional CDF of the available funds of V is

$$F_2(A_2|s(T,2)) = \begin{cases} 0 & A_2 < 0 \\ \dfrac{F_2(A_2|S(T,1)) - F_2(0|S(T,1))}{1-F_2(0|S(T,1))} & A_2 \geq 0 \end{cases} \quad (A.11)$$

The funds of V available at the end of n cycles of T days and the probability of bankruptcy after this period, with n >2, are calculated iteratively.

Given $n \geq 2$, a_n and $f_n(A_n)$, the funds of V available at the end of the cycle n+1 are $a_{n+1} = z_{n+1} + a_n - k\cdot T$, where z_{n+1} is the income of V generated during the cycle n+1, and its CDF is

$$F_{n+1}(A_{n+1}|S(T,n)) = \int_0^{F+n\cdot(z_M-k\cdot T)} F_{V,T}(A_{n+1}+k\cdot T-x)\cdot\frac{f_n(x)}{1-F_n(0)}dx + F_{V,T}(A_{n+1}+k\cdot T-F-n\cdot(z_M-k\cdot T))\cdot\frac{(1-H(q))^{n\cdot T}}{\prod_{i=1}^n(1-F_i(0))} \quad (A.12)$$

For all A_{n+1}. We note that $F_{n+1}(A_{n+1}|S(T,n)) = 0$ for $A_{n+1} \leq z_m - k\cdot T$, $F_{n+1}(A_{n+1}|S(T,n)) = 1$ for $A_{n+1} \geq F + (n+1)\cdot(z_M - k\cdot T)$, and the function is continuous for any value of A_{n+1} except $F + (n+1)\cdot(z_M - k\cdot T)$ and the size of the jump at that point is

$$\frac{(1-H(q))^{(n+1)\cdot T}}{\prod_{i=1}^n(1-F_i(0))} \quad (A.13)$$

Let us define

$$f_{n+1}(A_{n+1}) \coloneqq \left.\frac{dF_{n+1}(x|S(T,n))}{dx}\right|_{x=A} = \int_0^{\min\{F+n\cdot(z_M-k\cdot T), A_{n+1}+k\cdot T-z_m\}} f_{V,T}(A_{n+1}+k\cdot T-x)\cdot\frac{f_n(x)}{1-F_n(0)}dx + f_{V,T}(A_{n+1}+k\cdot T-F-n\cdot(z_M-k\cdot T))\cdot\frac{(1-H(q))^{n\cdot T}}{\prod_{i=1}^n(1-F_i(0))} \quad (A.14)$$

For $A_{n+1} < z_M - k\cdot T$.

$$f_{n+1}(A_{n+1}) \coloneqq \left.\frac{dF_{n+1}(x|S(T,n))}{dx}\right|_{x=A} = (1-H(q))^T \cdot \frac{f_n(A_{n+1}+k\cdot T-Z_M)}{1-F_n(0)} \quad (A.15)$$

$$+\int_{A_{n+1}+k\cdot T-z_M}^{\min\{F+n\cdot(z_M-k\cdot T),A_{n+1}+k\cdot T-z_m\}} f_{V,T}(A_{n+1}+k\cdot T-x)\cdot\frac{f_n(x)}{1-F_n(0)}dx$$
$$+f_{V,T}(A_{n+1}+k\cdot T-F-n\cdot(z_M-k\cdot T))\cdot\frac{(1-H(q))^{n\cdot T}}{\prod_{i=1}^{n}(1-F_i(0))}$$

For $z_M - k\cdot T \le A_{n+1} < F + (n+1)\cdot(z_M - k\cdot T)$ and for $A_{n+1} = F + (n+1)\cdot(z_M - k\cdot T)$, we extend the function by continuity.

The non-survival probability of V at the end of the cycle $n+1$ is $\bar{s}(T,n+1) = s(T,n)\cdot F_{n+1}(0\,|\,S(T,n))$ and the survival probability, $s(T,n+1) = s(T,n)\cdot(1-F_{n+1}(0\,|\,S(T,n)))$. In case of survival at the end of the cycle $n+1$, the conditional CDF of the available funds of V is

$$F_{n+1}(A_{n+1}|S(T,n+1)) = \begin{cases} 0 & A_{n+1}<0 \\ \dfrac{F_{n+1}(A_{n+1}|S(T,n+1))-F_{n+1}(0|S(T,n+1))}{1-F_{n+1}(0|S(T,n+1))} & A_{n+1}\ge 0 \end{cases} \tag{A.16}$$

In the case that x is a continuous uniform random variable with values between x_m and x_M,

$$H(X) = \begin{cases} 0 & X<x_m \\ \dfrac{X-x_m}{x_M-x_m} & x_m\le X\le x_M \\ 1 & X>x_{max} \end{cases} \tag{A.17}$$

$$F_V(Y) = \begin{cases} 0 & Y<(p-r)\cdot x_m-(v-r)\cdot q \\ \dfrac{Y-(p-r)\cdot x_m+(v-r)\cdot q}{(p-r)\cdot(x_M-x_m)} & (p-r)\cdot x_m-(v-r)\cdot q\le Y\le (p-v)\cdot q \\ 1 & Y>(p-v)\cdot q \end{cases} \tag{A.18}$$

$$F_{V<}(Y) = \begin{cases} 0 & Y<(p-r)\cdot x_m-(v-r)\cdot q \\ \dfrac{Y-(p-r)\cdot x_m+(v-r)\cdot q}{(p-r)\cdot(q-x_m)} & (p-r)\cdot x_m-(v-r)\cdot q\le Y< (p-v)\cdot q \\ 1 & Y\ge(p-v)\cdot q \end{cases} \tag{A.19}$$

$$F_{V,T}(Z) = \begin{cases} \displaystyle\sum_{t=1}^{T} p\left(\sum_{i=1}^{t}u_i\le t+\frac{Z-T\cdot(p-v)\cdot q}{(p-r)\cdot(q-x_m)}\right)\cdot\binom{T}{t}\cdot\left(\frac{q-x_m}{x_M-x_m}\right)^{t}\cdot\left(1-\frac{q-x_m}{x_M-x_m}\right)^{T-t} & Z<T\cdot(p-v)\cdot q \\ 1 & Z\ge T\cdot(p-v)\cdot q \end{cases} \tag{A.20}$$

Where u_i are continuous random variables with values between 0 and 1.

Using the notation $\bar{c}=\frac{c}{p}$, $\bar{r}=\frac{r}{p}$, $\bar{v}=\frac{v}{p}$, $\bar{q}=\frac{q-x_m}{x_M-x_m}$, $\bar{x}_m=\frac{x_m}{x_M-x_m}$, $\bar{Z}=\frac{Z-T\cdot(p-v)\cdot x_m}{p\cdot(x_M-x_m)}$ and $\bar{F}_{V,T}(\bar{Z}) = F_{V,T}(\bar{Z}\cdot p\cdot(x_M-x_m)+T\cdot(p-v)\cdot x_m)$, we have

$$\bar{F}_{V,T}(\bar{Z}) = \begin{cases} \displaystyle\sum_{t=0}^{T} F_{U,t}\left(t+\frac{\bar{Z}-T\cdot(1-\bar{v})\cdot\bar{q}}{(1-\bar{r})\cdot\bar{q}}\right)\cdot\binom{T}{t}\cdot\bar{q}^{t}\cdot(1-\bar{q})^{T-t} & \bar{Z}<T\cdot(1-\bar{v})\cdot\bar{q} \\ 1 & \bar{Z}\ge T\cdot(1-\bar{v})\cdot\bar{q} \end{cases} \tag{A.21}$$

$$\bar{f}_{V,T}(\bar{Z}) = \frac{1}{(1-\bar{r})\cdot\bar{q}}\sum_{t=0}^{T} f_{U,t}\left(t+\frac{\bar{Z}-T\cdot(1-\bar{v})\cdot\bar{q}}{(1-\bar{r})\cdot\bar{q}}\right)\cdot\binom{T}{t}\cdot\bar{q}^{t}\cdot(1-\bar{q})^{T-t} \quad \text{(A.22)}$$

Where $\bar{Z} \leq T\cdot(1-\bar{v})\cdot\bar{q}$.

$F_{U,t}$ i $f_{U,t}$ can be calculated from the Irwin-Hall distribution:

$$F_{U,n}(x) = \frac{1}{n!}\sum_{k=0}^{floor\ x}(-1)^k\binom{n}{k}(x-k)_+^n \quad \text{(A.23)}$$

$$f_{U,n}(x) = \frac{1}{(n-1)!}\sum_{k=0}^{floor\ x}(-1)^k\binom{n}{k}(x-k)_+^{n-1} \quad \text{(A.24)}$$

where $(x-k)_+ = \max(0,x-k)$.

From $\bar{F}_{V,T}(\bar{Z})$ and $\bar{f}_{V,T}(\bar{Z})$ it could be calculated $F_{V,T}(Z)$ and $f_{V,T}(Z)$ with

$$F_{V,T}(Z) = \bar{F}_{V,T}\left(\frac{Z-T\cdot(p-v)\cdot x_m}{p\cdot(x_M-x_m)}\right) \quad \text{(A.25)}$$

$$f_{V,T}(Z) = \frac{1}{p\cdot(x_M-x_m)}\cdot\bar{f}_{V,T}\left(\frac{Z-T\cdot(p-v)\cdot x_m}{p\cdot(x_M-x_m)}\right) \quad \text{(A.26)}$$